

# Advanced Systems Theory

25/01/2024, Thursday, 8:30 – 10:30

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## 1 Disturbance decoupling with measurement feedback

(15 + 10 = 25 pts)

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Consider the system

$$\begin{aligned}\dot{x} &= Ax + Bu + Ed \\ y &= Cx \\ z &= Hx\end{aligned}$$

with

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = [0 \quad 0 \quad 1], \quad H = [0 \quad 1 \quad a]$$

where  $a$  is a real number.

- Determine all values of  $a$  such that the problem of disturbance decoupling (from  $d$  to  $z$ ) by measurement feedback is solvable.
- For those values of  $a$ , compute a dynamic controller that makes the system from  $d$  to  $z$  decoupled.

## 2 Lyapunov equation

(10 + 10 = 20 pts)

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Let  $Q = [q_{ij}]$  be an  $n \times n$  real symmetric matrix and let  $d_i$  with  $1 \leq i \leq n$  be positive numbers. Define  $P = [p_{ij}]$  by

$$p_{ij} = \frac{q_{ij}}{d_i + d_j}$$

where  $1 \leq i, j \leq n$ .

- Find an  $n \times n$  diagonal matrix  $D$  such that  $DP + PD = Q$ .
- Show that  $P$  is positive semidefinite if so is  $Q$ .

## 3 Synchronization

(25 pts)

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Consider the multi-agent system given by the communication graph  $G$  agent dynamics

$$\dot{x}_i = Ax_i + Bz_i$$

and the diffusive coupling

$$z_i = K \sum_{j \in N(i)} (x_j - x_i)$$

where  $i \in \{1, 2, \dots, p\}$ . Suppose that the graph  $G$  is connected and the smallest nonzero eigenvalue of its Laplacian is  $\lambda_2$ . Find  $K$  such that the multi-agent system is synchronized by solving the algebraic Riccati equation (via Hamiltonian)

$$A^T P + PA - 2\lambda_2 P B B^T P + Q = 0$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \lambda_2 = \frac{1}{2}, \quad \text{and} \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Consider the true unknown system

$$\mathbf{x}(t+1) = A_{\text{true}}\mathbf{x}(t) + B_{\text{true}}\mathbf{u}(t)$$

where  $t \in \mathbb{Z}_+$ ,  $\mathbf{x} \in \mathbb{R}^n$ , and  $\mathbf{u} \in \mathbb{R}^m$ . Suppose that the data

$$U_- = [u(0) \quad u(1) \quad \cdots \quad u(T-1)] \quad \text{and} \quad X = [x(0) \quad x(1) \quad \cdots \quad x(T)]$$

are harvested from the true system. Define

$$X_- := [x(0) \quad x(1) \quad \cdots \quad x(T-1)] \quad \text{and} \quad X_+ := [x(1) \quad x(1) \quad \cdots \quad x(T)].$$

Also, define the set of all explaining systems

$$\Sigma := \{(A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \mid X_+ = AX_- + BU_-\}.$$

The data  $(U_-, X)$  are said to be *informative for stability* if  $A$  is stable for all  $A$  such that  $(A, B) \in \Sigma$  for some  $B$ . Show that the data  $(U_-, X)$  are informative for stability if and only if  $X_-$  has full row rank and there exists a right inverse  $X_-^\dagger$  of  $X_-$  such that  $X_+X_-^\dagger$  is stable and  $U_-X_-^\dagger = 0$ .

HINT: For the ‘only if’ part, try first to show that  $X_-$  has full row rank and

$$\ker [X_-^T \quad U_-^T] \subseteq \ker [I_n \quad 0_{n,m}].$$

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10 pts free